

Finite-dimensional representations of difference operators, and the identification of remarkable matrices

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Abstract

Two square matrices of (arbitrary) order N are introduced. They are defined in terms of N arbitrary numbers z_n , and of an arbitrary additional parameter (a respectively q), and provide finite-dimensional representations of the two operators acting on a function $f(z)$ as follows: $[f(z+a) - f(z)]/a$ respectively $[f(qz) - f(z)]/[(q-1)z]$. These representations are exact—in a sense explained in the paper—when the function $f(z)$ is a polynomial in z of degree less than N . This formalism allows to transform difference equations valid in the space of polynomials of degree less than N into corresponding matrix-vector equations. As an application of this technique several remarkable square matrices of order N are identified, which feature explicitly N arbitrary numbers z_n , or the N zeros of polynomials belonging to the Askey and q -Askey schemes. Several of these findings have a Diophantine character.

1 Introduction

This paper is focussed on $(N \times N)$ -matrices which provide finite-dimensional representations of difference operators yielding *exact* results in the context of the functional space spanned by polynomials of degree less than N ; the precise meaning of this statement is clarified below.

These findings extend to *difference* operators the results reported for the standard *differential* operator in Section 2.4 (entitled "Finite dimensional representations of differential operators, Lagrangian interpolation, and all that") of [1] (and see also papers referred to there). Let us summarize here—for completeness, and also to introduce some notation used throughout—the essence of those findings.

Notation 1.1. Throughout this paper N is an arbitrary positive integer (unless otherwise explicitly indicated); N -vectors are denoted by underlined (Latin or Greek) letters, so that, for instance, the N vector \underline{v} has the N components v_n ; likewise $(N \times N)$ -matrices are denoted by twice-underlined (Latin or Greek) letters, so that, for instance, the $(N \times N)$ -matrix $\underline{\underline{M}}$ features the N^2 components M_{nm} . Here and throughout the indices n, m, ℓ run over the integers from 1 to N , unless otherwise indicated. Attention is generally restricted to functions $f(z)$ which depend *analytically* on their argument z , and in particular that are polynomials in their argument z . The formulas written below are generally valid for arbitrary, complex values of all the quantities denoted by (Latin or Greek) letters, up to obvious limitations for cases when limits might have to be taken: for instance $g(a, z) = [f(a + z) - f(z)]/a$ has a clear significance for every value of the quantity a except for $a = 0$, but it also clearly implies $g(0, z) = df(z)/dz$. Finally, we use throughout the notation \mathbf{i} to denote the *imaginary unit*, so that $\mathbf{i}^2 = -1$. \square

To summarize the previous results [1] let us assume that the function $f(z)$ is a polynomial of degree less than N ,

$$f(z) = \sum_{m=1}^N [c_m z^{N-m}] , \quad (1)$$

and let us then express it as a linear combination—with coefficients f_n —of the N interpolational polynomials $p_{N-1}^{(n)}(z)$ —all of them of degree $N-1$ in z —defined as follows in terms of the N , *arbitrarily assigned*, numbers z_n :

$$p_{N-1}^{(n)}(z) \equiv p_{N-1}^{(n)}(z; \underline{z}) = \prod_{\ell=1, \ell \neq n}^N \left(\frac{z - z_\ell}{z_n - z_\ell} \right) , \quad (2)$$

$$f(z) = \sum_{n=1}^N \left[f_n p_{N-1}^{(n)}(z) \right] . \quad (3)$$

This definition, (2), of the interpolational polynomials $p_{N-1}^{(n)}(z)$ clearly implies the relation

$$p_{N-1}^{(n)}(z_m) \equiv p_{N-1}^{(n)}(z_m; \underline{z}) = \delta_{nm} , \quad n, m = 1, \dots, N ; \quad (4)$$

hence the N coefficients f_n in the right-hand side of (3) are just the values of $f(z)$ at the N points z_n :

$$f_n = f(z_n) \ , \quad n = 1, \dots, N \ . \quad (5)$$

The above formulas correspond to the standard formulation of Lagrangian interpolation: the N interpolational points z_n can be arbitrarily assigned, except for the restriction that they be all different among themselves, see (2) (otherwise some limits would have to be taken). They entail a one-to-one relationship among the N -vector \underline{f} featuring the N components f_n ,

$$\underline{f} = (f_1, \dots, f_N) \ , \quad (6)$$

and the function $f(z)$ restricted to be a polynomial in z of degree less than N .

In Section 2.4 of [1] certain relations are reported among *differential* operators acting on such functions $f(z)$, see (1), and appropriately defined $(N \times N)$ -matrices acting on the N -vector \underline{f} . The basic formula of this kind, corresponding to the definition

$$f^{(r)}(z) = \left(\frac{d}{dz} \right)^r f(z) \ , \quad r = 0, 1, 2, \dots \ , \quad (7)$$

reads (see, up to minor notational changes, eq. (2.4.1-9) of [1])

$$\underline{f}^{(r)} = (\underline{V} \underline{D} \underline{V}^{-1})^r \underline{f} = \underline{V} (\underline{D})^r \underline{V}^{-1} \underline{f} = \underline{V} (\underline{D})^r \underline{f}(\underline{Z}) \underline{w} \ , \quad r = 0, 1, 2, \dots \ . \quad (8)$$

Here the N components $f_n^{(r)}$ of the N -vector $\underline{f}^{(r)}$ are of course the N values $f^{(r)}(z_n)$ that the r -th derivative $f^{(r)}(z)$ of the function $f(z)$ (see (7)) takes at the N interpolational points z_n ,

$$f_n^{(r)} \equiv (\underline{f}^{(r)})_n = f^{(r)}(z_n) \ , \quad n = 1, \dots, N \ , \quad r = 0, 1, 2, \dots \quad (9a)$$

or, equivalently,

$$\underline{f}^{(r)} = f^{(r)}(\underline{Z}) \underline{u} \ , \quad (9b)$$

with the N -vector \underline{u} having all elements equal to *unity*,

$$\underline{u} = (1, 1, \dots, 1) \ ; \quad u_n = 1 \ , \quad n = 1, \dots, N \ . \quad (10)$$

The $(N \times N)$ -matrices $\underline{Z} \equiv \underline{Z}(\underline{z})$, $\underline{D} \equiv \underline{D}(\underline{z})$, $\underline{V} \equiv \underline{V}(\underline{z})$ and the N -vector $\underline{w} \equiv \underline{w}(\underline{z})$ are defined componentwise as follows, in terms of the N arbitrary numbers z_n which are the N components of the N -vector \underline{z} :

$$\underline{Z}(\underline{z}) = \text{diag}[z_n] \ ; \quad Z_{nm} = \delta_{nm} z_n \ ; \quad (11)$$

$$D_{nn} \equiv D_{nn}(\underline{z}) = \sum_{\ell=1, \ell \neq n}^N \left(\frac{1}{z_n - z_\ell} \right) \ , \quad n = 1, \dots, N \ , \quad (12a)$$

$$D_{nm} \equiv D_{nm}(\underline{z}) = \left(\frac{1}{z_n - z_m} \right), \quad n \neq m, \quad n, m = 1, \dots, N; \quad (12b)$$

$$V_{nm} \equiv V_{nm}(\underline{z}) = \delta_{nm} \prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell); \quad (13)$$

$$\underline{w} = \underline{V}^{-1} \underline{u}, \quad w_n \equiv w_n(\underline{z}) = \prod_{\ell=1, \ell \neq n}^N \left(\frac{1}{z_n - z_\ell} \right). \quad (14)$$

These formulas imply [1] that, whenever a function $f(z)$ which is a polynomial of degree less than N in z , see (1), satisfies the differential equation (with no *a priori* restriction on the positive integer R and on the $R+2$ functions $d_r(z)$ and $g(z)$)

$$\sum_{r=0}^R \left[d_r(z) \left(\frac{d}{dz} \right)^r f(z) \right] = g(z), \quad (15a)$$

the following N -vector equation is as well valid:

$$\sum_{r=0}^R [d_r(\underline{Z}) (\underline{D})^r] f(\underline{Z}) \underline{w} = g(\underline{Z}) \underline{w}; \quad (15b)$$

with the $(N \times N)$ -matrices \underline{Z} and \underline{D} and the N -vector \underline{w} defined as above, see (11), (12) and (14), in terms of N numbers z_n , arbitrary except for the condition that they be all different among themselves. With remarkable consequences [1].

The main findings of this paper extend to *difference* operators these results; they are reported in the next Section 2 and proven in Section 4. In Section 3, as examples of applications of these findings several $(N \times N)$ -matrices displaying remarkable features are explicitly defined in terms of a few arbitrary parameters and in addition of N arbitrary numbers z_n or, alternatively, of the N zeros \bar{z}_n of the polynomials of degree N belonging to the Askey and q -Askey schemes. A terse Section 5 entitled Outlook outlines possible future developments. Some calculations are confined to an Appendix A in order to avoid unessential interruptions in the flow of the presentation.

2 Main results

Let the two difference operators $\hat{\nabla}(a)$ and $\check{\nabla}(q)$ be defined as follows:

$$\hat{\nabla}(a) f(z) = \frac{f(z+a) - f(z)}{a}, \quad (16)$$

$$\check{\nabla}(q) f(z) = \frac{f(z) - f(qz)}{(1-q)z}. \quad (17)$$

It is plain that the first of these difference operators becomes the standard differential operator d/dz as $a \rightarrow 0$, and likewise the second as $q \rightarrow 1$:

$$\hat{\nabla}(0) f(z) = \check{\nabla}(1) f(z) = f'(z) . \quad (18)$$

But hereafter we assume for simplicity that $a \neq 0$ and $q \neq 1$.

It is also plain that for the difference operator $\hat{\nabla}(a)$ there holds the following eigenvalue equation:

$$z \hat{\nabla}(a) \hat{f}_k(z; a) = k \hat{f}_k(z; a) , \quad k = 0, 1, 2, \dots , \quad (19a)$$

with the eigenfunction $\hat{f}_k(z; a)$ coinciding—up to an irrelevant multiplicative constant—with the "shifted-factorial" $(z, a)_k$,

$$\hat{f}_k(z; a) = (z, a)_k , \quad (19b)$$

itself defined (here and hereafter) as follows:

$$(z, a)_0 = 1 ; \quad (z, a)_r = \prod_{s=0}^{r-1} (z + s a) , \quad r = 1, 2, \dots . \quad (19c)$$

Likewise for the difference operator $\check{\nabla}(q)$ there holds the following eigenvalue equation:

$$z \check{\nabla}(q) \check{f}_k(z) = \frac{1 - q^k}{1 - q} \check{f}_k(z) , \quad \check{f}_k(z) = z^k . \quad (20a)$$

Remark 2.1. In the eigenvalue equation (19) the *nonnegative integer* eigenvalues k are independent of the parameter a , while the corresponding eigenfunctions $\hat{f}_k(z; a) = (z, a)_k$ depend on both k and a . Viceversa, in the eigenvalue equation (20a) the eigenfunctions $\check{f}_k(z) = z^k$ do not depend on the parameter q , while the corresponding eigenvalues $(1 - q^k) / (1 - q)$ depend on both k and the parameter q ; in this case k is *a priori* not restricted to take *integer*, or even *real*, values, but in our treatment we will in fact restrict attention also in this case to *nonnegative integer* values of the parameter k , so that the eigenfunction $\check{f}_k(z) = z^k$ is holomorphic, and the eigenvalues in (20a) read as follows:

$$\begin{aligned} \frac{1 - q^k}{1 - q} &= 0 \quad \text{for } k = 0 , \\ \frac{1 - q^k}{1 - q} &= 1 + q + q^2 + \dots + q^{k-1} = \sum_{s=0}^{k-1} (q^s) \quad \text{for } k = 1, 2, 3, \dots \quad \square . \end{aligned} \quad (20b)$$

Notation 2.1. It is convenient to also introduce the simpler operators $\hat{\delta}(a)$ and $\check{\delta}(q)$ defined as follows:

$$\hat{\delta}(a) f(z) = f(z + a) , \quad \check{\delta}(q) f(z) = f(q z) . \quad (21a)$$

This implies of course that the operators $\hat{\nabla}(a)$ and $\check{\nabla}(q)$ defined above are related to these operators—acting on functions $f(z)$ of the variable z —as follows:

$$\hat{\nabla}(a) = a^{-1} \left[\hat{\delta}(a) - 1 \right] , \quad \check{\nabla}(q) = [(1-q) z]^{-1} [1 - \check{\delta}(q)] . \quad \square \quad (21b)$$

The main result of this paper is the identification of two $(N \times N)$ -matrices $\hat{\underline{\underline{\delta}}}(a; \underline{z})$ and $\check{\underline{\underline{\delta}}}(q; \underline{z})$ which provide—together with the $N \times N$ diagonal matrix $\underline{\underline{Z}} = \text{diag}[z_n]$, see (11)—finite-dimensional representations of the two operators $\hat{\delta}(a)$ respectively $\check{\delta}(q)$. These matrices are defined componentwise as follows:

$$\left(\hat{\underline{\underline{\delta}}}(a; \underline{z}) \right)_{nm} = \prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n + a - z_\ell}{z_m - z_\ell} \right) ; \quad (22a)$$

$$\left(\check{\underline{\underline{\delta}}}(q; \underline{z}) \right)_{nm} = \prod_{\ell=1, \ell \neq m}^N \left(\frac{q z_n - z_\ell}{z_m - z_\ell} \right) . \quad (22b)$$

And of course the $(N \times N)$ -matrices $\hat{\underline{\underline{\nabla}}}(a; \underline{z}) = [\hat{\underline{\underline{\delta}}}(a; \underline{z}) - 1] / a$ respectively $\check{\underline{\underline{\nabla}}}(q; \underline{z}) = [(1-q) \underline{\underline{Z}}(\underline{z})]^{-1} [1 - \check{\underline{\underline{\delta}}}(q; \underline{z})]$ (see (21b) and (22)) which provide—again, together with the $N \times N$ diagonal matrix $\underline{\underline{Z}}$ —finite-dimensional representations of the two difference operators $\hat{\nabla}(a)$ respectively $\check{\nabla}(q)$ are correspondingly defined componentwise as follows:

$$\left(\hat{\underline{\underline{\nabla}}}(a; \underline{z}) \right)_{nm} = a^{-1} \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n + a - z_\ell}{z_m - z_\ell} \right) \right] - \delta_{nm} \right\} ; \quad (23a)$$

$$\left(\check{\underline{\underline{\nabla}}}(q; \underline{z}) \right)_{nm} = [(1-q) z_n]^{-1} \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{q z_n - z_\ell}{z_m - z_\ell} \right) \right] - \delta_{nm} \right\} . \quad (23b)$$

Notation 2.2. In all the above formulas and hereafter δ_{nm} is the standard Kronecker symbol,

$$\delta_{nm} = 1 \quad \text{if } n = m , \quad \delta_{nm} = 0 \quad \text{if } n \neq m ; \quad (24)$$

while—unless otherwise explicitly specified—the N numbers z_n are *arbitrary* (but obviously all different among themselves, and the *same* set of N numbers throughout). Let us reiterate that all the $(N \times N)$ -matrices introduced above are defined in terms of these N *a priori* arbitrary numbers, i. e. of the N components of the N -vector \underline{z} . This important fact should always be kept in mind, even though, for notational simplicity, we occasionally omit below to indicate explicitly this dependence. \square

The significance of the statement that the $(N \times N)$ -matrices $\hat{\underline{\underline{\delta}}}(a; \underline{z})$, $\check{\underline{\underline{\delta}}}(q; \underline{z})$, $\hat{\underline{\underline{\nabla}}}(a; \underline{z})$ respectively $\check{\underline{\underline{\nabla}}}(q; \underline{z})$ provide finite-dimensional representations of the

corresponding operators $\hat{\delta}(a)$, $\check{\delta}(q)$, $\hat{\nabla}(a)$ respectively $\check{\nabla}(q)$ —and, most importantly, that these representations are *exact* in the functional space spanned by polynomials of degree less than N —is made explicit by the following properties which extend to these operators results reported in Section 2.4 of [1] for *differential* operators (as tersely summarized in the preceding Section 1). The main idea is again to identify relations among operators acting on functions $f(z)$ —always restricted to live in the functional space spanned by polynomials in z of degree less than N , see (1)—and appropriately defined matrices acting on the corresponding N -vector \underline{f} , see (11), (22) and (23).

We report in the following Sections 2.1 respectively 2.2 the main relevant formulas for the operators $\hat{\delta}(a)$ and $\hat{\Delta}(a)$ respectively $\check{\delta}(q)$ and $\check{\Delta}(q)$; proofs of those of these results that are not immediately obvious are postponed to Section 4. And in Appendix B we report a few additional *remarkable* properties of the two $(N \times N)$ -matrices $\underline{\hat{\delta}}(a; \underline{z})$ and $\underline{\hat{\Delta}}(q; \underline{z})$.

2.1 The $(N \times N)$ -matrices $\underline{\hat{\delta}}(a)$ and $\underline{\hat{\nabla}}(a)$

In this Section 2.1 we report the main results concerning finite-dimensional representations of the operators $\hat{\delta}(a)$ and $\hat{\nabla}(a)$, see (16), (21a) and (21b).

Lemma 2.1.1. Let $f(z)$ be an arbitrary polynomial in z of degree less than N , see (1), and let us denote with the notation $\hat{f}^{[a,r]}(z)$ the polynomial (also of degree less than N) that obtains by applying to it r times the operator $\hat{\delta}(a)$, see (21a):

$$\begin{aligned} \hat{f}^{[a,r]}(z) &\equiv \left(\hat{\delta}(a) \right)^r f(z) = f(z + r a) \\ &= \sum_{m=1}^N \left[c_m (z + r a)^{N-m} \right] = \hat{f}^{[ar]}(z) \ , \quad r = 1, 2, \dots \end{aligned} \quad (25a)$$

Now associate to $f(z)$ respectively to $\hat{f}^{[ar]}(z)$ the N -vectors $\underline{f} \equiv \underline{f}(\underline{z})$ respectively $\underline{\hat{f}}^{[ar]} \equiv \underline{\hat{f}}^{[ar]}(\underline{z})$, whose N components f_n respectively $\hat{f}_n^{[ar]}$ are the N values that the polynomials (of degree less than N , see (1)) $f(z)$ respectively $\hat{f}^{[ar]}(z) = f(z + r a)$ take at the N (arbitrary) points z_n ,

$$\begin{aligned} f_n &= f(z_n) \ , \quad \hat{f}_n^{[ar]} = \hat{f}^{[ar]}(z_n) = f(z_n + a r) \ , \\ n &= 1, \dots, N \ ; \quad r = 0, 1, 2, \dots \end{aligned} \quad (25b)$$

There holds then the N -vector formula

$$\underline{\hat{f}}^{[ar]} = \left[\underline{\hat{\delta}}(a; \underline{z}) \right]^r \underline{f} \ , \quad r = 0, 1, 2, \dots \ , \quad (26)$$

with the $(N \times N)$ -matrices $\underline{\hat{\delta}}(a; \underline{z})$ defined componentwise by (22a). \square

Remark 2.1.1. The fact that the matrix $\left[\underline{\hat{\delta}}(a; \underline{z}) \right]^r$ appearing in the right-hand side of the last equation depends—consistently with the left-hand side of

this equation—on the product ar (rather than separately on a and r) is not immediately obvious from its definition (22a), but is in fact true, indeed see below *Remark 4.1.1*. \square

Clearly this finding implies an analogous result for the *difference* operator $\hat{\nabla}(a)$, see (21b):

Lemma 2.1.2. Let $f(z)$ be an arbitrary polynomial in z of degree less than N , see (1), and let us denote with the notation $\hat{f}^{[[a,r]]}(z)$ the polynomial that obtains by applying to it r times the difference operator $\hat{\nabla}(a)$, see (16) and (21b):

$$\hat{f}^{[[a,r]]}(z) \equiv \left(\hat{\nabla}(a) \right)^r f(z) = \left[\frac{\hat{\delta}(a) - 1}{a} \right]^r f(z) , \quad r = 0, 1, 2, \dots \quad (27a)$$

Now associate to $f(z)$ respectively to $\hat{f}^{[[a,r]]}(z)$ the N -vectors $\underline{f} \equiv \underline{f}(\underline{z})$ respectively $\underline{\hat{f}}^{[[a,r]]} \equiv \underline{\hat{f}}^{[[a,r]]}(\underline{z})$, whose N components $f_n \equiv f_n(\underline{z})$ respectively $\hat{f}_n^{[[a,r]]} \equiv \hat{f}_n^{[[a,r]]}(\underline{z})$ are the N values that the polynomials $f(z)$ respectively $\hat{f}^{[[a,r]]}(z)$ take at the N (arbitrary) points z_n ,

$$f_n = f(z_n) , \quad \hat{f}_n^{[[a,r]]} = \hat{f}^{[[a,r]]}(z_n) , \quad n = 1, \dots, N . \quad (27b)$$

There holds then the N -vector formula

$$\underline{\hat{f}}^{[[a,r]]} = \left[\underline{\hat{\nabla}}(a; \underline{z}) \right]^r \underline{f} , \quad r = 0, 1, 2, \dots , \quad (28)$$

of course with the $(N \times N)$ -matrix $\underline{\hat{\nabla}}(a; \underline{z})$ defined componentwise by (23a). \square

Remark 2.1.2. It is plain that the operator $\hat{\nabla}(a)$, when acting on a polynomial in z of degree M , yields a polynomial of degree $M - 1$; hence, when it acts r times on any polynomial of degree less than N it yields an identically vanishing result if the integer r equals or exceeds N . Hence the right-hand side of (27a) vanishes for $r \geq N$, and this implies that the $(N \times N)$ -matrix $\underline{\hat{\nabla}}(a; \underline{z})$ features the *remarkable* property

$$\left[\underline{\hat{\nabla}}(a; \underline{z}) \right]^N = \underline{0} , \quad (29)$$

where $\underline{0}$ denotes of course the $(N \times N)$ -matrix with all elements vanishing. \square

The following *Proposition* and *Corollaries* are immediate consequences of these findings.

Proposition 2.1.1. Let the difference operator $\hat{D}(a)$ be defined as follows,

$$\hat{D}(a) = \sum_{r=0}^R \left\{ \hat{d}_r(z) \left[\hat{\delta}(a) \right]^r \right\} , \quad (30)$$

where the positive integer R and the $R+1$ functions $\hat{d}_r(z)$ are *a priori* arbitrary (but see below the restriction on the function $f(z)$), and let

$$\hat{D}(a) f(z) = g(z) \quad (31)$$

with $f(z)$ a polynomial in z of degree less than N , see (1) (but note: no such condition on $g(z)$). There then holds the N -vector equation

$$\underline{\underline{\hat{D}}}(a; \underline{z}) \underline{f}(\underline{z}) = \underline{g}(\underline{z}) \quad (32)$$

with the $(N \times N)$ -matrix $\underline{\underline{\hat{D}}}(a; \underline{z})$ defined as follows,

$$\underline{\underline{\hat{D}}}(a; \underline{z}) = \sum_{r=0}^R \left\{ \hat{d}_r(\underline{Z}) \left[\underline{\hat{\delta}}(a; \underline{z}) \right]^r \right\} , \quad (33)$$

and of course the N -vectors $\underline{f}(\underline{z})$ and $\underline{g}(\underline{z})$ defined so that their N components are

$$(\underline{f}(\underline{z}))_n = f(z_n) , \quad (\underline{g}(\underline{z}))_n = g(z_n) , \quad n = 1, \dots, N , \quad (34a)$$

or, equivalently,

$$\underline{f}(\underline{z}) = f(\underline{Z}) \underline{u} , \quad \underline{g}(\underline{z}) = g(\underline{Z}) \underline{u} , \quad (34b)$$

of course with the $(N \times N)$ -matrices $\underline{\underline{\hat{D}}}(a; \underline{z})$ respectively \underline{Z} defined component-wise by (22a) respectively (11) and the "unit" N -vector \underline{u} defined by (10). \square

Corollary 2.1.1. If in (31) $g(z) = 0$, i. e. if for the operator $\hat{D}(a)$, see (30), there holds the equation

$$\hat{D}(a) f(z) = 0 , \quad (35a)$$

with $f(z)$ a polynomial in z of degree less than N , see (1), then the $(N \times N)$ -matrix $\underline{\underline{\hat{D}}}(a; \underline{z})$, see (33), has vanishing determinant,

$$\det \left[\underline{\underline{\hat{D}}}(a; \underline{z}) \right] = 0 . \quad \square \quad (35b)$$

Corollary 2.1.2. If the operator $\hat{D}(a)$, see (30), has the eigenvalue b ,

$$\hat{D}(a) f^{(b)}(z) = b f^{(b)}(z) \quad (36a)$$

with the corresponding eigenfunction $f^{(b)}(z)$ being a polynomial in z of degree less than N , see (1), then the corresponding $(N \times N)$ -matrix $\underline{\underline{\hat{D}}}(a; \underline{z})$, see (33), features the same eigenvalue b ,

$$\underline{\underline{\hat{D}}}(a; \underline{z}) \underline{f}^{(b)} = b \underline{f}^{(b)} , \quad (36b)$$

and the corresponding eigenvector $\underline{f}^{(b)}$ is given by the following simple prescription,

$$\underline{f}^{(b)} = f^{(b)}(\underline{Z}) \underline{u} ; \quad (\underline{f}^{(b)})_n = f^{(b)}(z_n) , \quad n = 1, \dots, N , \quad (36c)$$

where of course the $(N \times N)$ -matrix \underline{Z} , respectively the N -vector \underline{u} , are again defined by (11) respectively (10). \square

Clearly these equations are merely examples of the neat prescriptions

$$\begin{aligned} \hat{d}_s(z) &\Rightarrow \hat{d}_s(\underline{Z}(\underline{z})) , & \hat{\delta}(a) &\Rightarrow \underline{\hat{\delta}}(a; \underline{z}) , & \hat{\nabla}(a) &\Rightarrow \underline{\hat{\nabla}}(a; \underline{z}) , \\ f(z) &\Rightarrow \underline{f}(\underline{z}) = f(\underline{Z}) \underline{u} , \end{aligned} \quad (37)$$

which allow to transform equations involving the action of the multiplicative operator $\hat{d}_s(z)$ (see (30)) and of the operators $\hat{\delta}(a)$ and $\hat{\nabla}(a)$ (see (21a) and (16) or (21b)) acting on functions $f(z)$, into corresponding equations involving the action of corresponding $(N \times N)$ -matrices on corresponding N -vectors; rules that involve the introduction of N arbitrary numbers z_n (all different among themselves), and that are applicable whenever these operators act on functions $f(z)$ which are polynomials in z of degree less than the arbitrary number N , and that involve the simultaneous replacement of the function $f(z)$ into the N -vector $\underline{f}(\underline{z})$ of components $f_n(\underline{z}) = f(z_n)$.

Remark 2.1.3. An interesting generalization of all the findings reported above (in this Section 2.1) is to the case in which the function $f(z)$, instead of being a polynomial of degree less than N in z , is a polynomial of degree less than N in a variable $\zeta \equiv \zeta(z)$,

$$f(\zeta) \equiv f(\zeta(z)) = \sum_{m=1}^N \left\{ c_m [\zeta(z)]^{N-m} \right\} . \quad (38)$$

It is then easily seen that all the results reported above (in this Section 2.1) remain valid, provided the $(N \times N)$ -matrix $\hat{\underline{\delta}}(a; \underline{z})$, see (22a), is replaced by the $(N \times N)$ -matrix $\tilde{\underline{\delta}}(a; \underline{z})$ defined componentwise as follows:

$$\left(\tilde{\underline{\delta}}(a; \underline{z}) \right)_{nm} = \prod_{\ell=1, \ell \neq m}^N \left[\frac{\zeta(z_n + a) - \zeta_\ell}{\zeta_m - \zeta_\ell} \right] , \quad n, m = 1, \dots, N , \quad (39a)$$

where of course

$$\zeta_n \equiv \zeta(z_n) , \quad \underline{\zeta} = (\zeta_1, \dots, \zeta_N) . \quad (39b)$$

Of course likewise the matrix $\hat{\underline{\nabla}}(a; \underline{z})$ is replaced by the matrix $\tilde{\underline{\nabla}}(a; \underline{z}) = [\tilde{\underline{\delta}}(a; \underline{z}) - 1]/a$, of components

$$\begin{aligned} \left(\tilde{\underline{\nabla}}(a; \underline{z}) \right)_{nm} &= a^{-1} \left\{ \prod_{\ell=1, \ell \neq m}^N \left[\frac{\zeta(z_n + a) - \zeta_\ell}{\zeta_m - \zeta_\ell} \right] - \delta_{nm} \right\} , \\ n, m &= 1, \dots, N ; \end{aligned} \quad (40)$$

and the N -vector $\underline{f}(\underline{z})$ of components $f_n(\underline{z}) = f(z_n)$ is replaced by the N -vector $\underline{f}(\underline{\zeta}(\underline{z}))$ of components $f_n(\underline{\zeta}(\underline{z})) = f(\zeta_n)$, while the N -vector $\underline{f}(\underline{z} + a \underline{u})$ of components $f_n(\underline{z} + a \underline{u}) = f(z_n + a)$ is replaced by the N -vector $\underline{f}(\underline{\zeta}(\underline{z} + a \underline{u}))$ of components $f_n(\underline{z} + a \underline{u}) = f(\zeta(z_n + a))$. \square

The proof of this *Remark 2.1.3* is quite analogous to that of *Lemma 2.1.1* (see Section 4) and is therefore omitted.

2.2 The $(N \times N)$ -matrices $\check{\underline{\delta}}(q; \underline{z})$ and $\check{\underline{\nabla}}(q; \underline{z})$

In this Section 2.2 we report the main results concerning finite-dimensional representations of the operators $\check{\delta}(q)$ and $\check{\nabla}(q)$, see (17), (21a) and (21b). It should be mentioned that a somewhat less explicit, but essentially equivalent, finite-dimensional representation of the difference operator $\check{\nabla}(q)$ was already provided almost two decades ago by Chakrabarti and Jagannathan [3]; and also, in the context of a more special case ("3-body problem") in [4].

Lemma 2.2.1. Let $f(z)$ be an arbitrary polynomial in z of degree less than N , see (1), and let us denote with the notation $\check{f}^{\{q,r\}}(z)$ the polynomial (clearly of the same degree) that obtains by applying to it r times the operator $\check{\delta}(q)$, see (21a):

$$\begin{aligned} \check{f}^{\{q,r\}}(z) &\equiv (\check{\delta}(q))^r f(z) = f(q^r z) = \sum_{m=1}^N \left[c_m (q^r z)^{N-m} \right] = \check{f}^{\{q^r\}}(z) , \\ r &= 1, 2, 3, \dots \end{aligned} \quad (41a)$$

Now associate to $f(z)$ respectively to $\check{f}^{\{q^r\}}(z)$ the N -vectors $\underline{f} \equiv \underline{f}(\underline{z})$ respectively $\underline{\check{f}}^{\{q^r\}} \equiv \underline{\check{f}}^{\{q^r\}}(\underline{z})$, whose N components f_n respectively $\check{f}_n^{\{q^r\}}$ are the N values that the polynomials $f(z)$ respectively $\check{f}^{\{q^r\}}(z) = f(q^r z)$ take at the N (arbitrary) points z_n ,

$$\begin{aligned} f_n &= f(z_n) , \quad \check{f}_n^{\{q^r\}} = \check{f}^{\{q^r\}}(z_n) = f(q^r z_n) , \\ n &= 1, \dots, N ; \quad r = 0, 1, 2, \dots \end{aligned} \quad (41b)$$

There holds then the N -vector formula

$$\underline{\check{f}}^{\{q^r\}} = [\check{\underline{\delta}}(q; \underline{z})]^r \underline{f} , \quad r = 0, 1, 2, \dots , \quad (42)$$

with the $(N \times N)$ -matrices $\check{\underline{\delta}}(q; \underline{z})$ defined componentwise by (22b). \square

Remark 2.2.1. The fact that the matrix $[\check{\underline{\delta}}(q; \underline{z})]^r$ appearing in the right-hand side of the last equation depends—consistently with the left-hand side of this equation—on the single quantity q^r (rather than separately on q and r) is not immediately obvious from its definition (22b), but is in fact true, indeed see below *Remark 4.2.1.* \square

Clearly this finding implies an analogous result for the *difference* operator $\check{\nabla}(q)$, see (21b):

Lemma 2.2.2. Let $f(z)$ be an arbitrary polynomial in z of degree less than N , see (1), and let us denote with the notation $\check{f}^{\{\{q,r\}\}}(z)$ the polynomial that obtains by applying to it r times the difference operator $\check{\nabla}(q)$, see (17) and (21b):

$$\check{f}^{\{\{q,r\}\}}(z) \equiv (\check{\nabla}(q))^r f(z) = \left[\frac{\check{\delta}(q) - 1}{(q - 1)z} \right]^r f(z) , \quad r = 0, 1, 2, \dots \quad (43a)$$

Now associate to $f(z)$ respectively to $\check{f}^{\{\{q,r\}\}}(z)$ the N -vectors $\underline{f} \equiv \underline{f}(\underline{z})$ respectively $\underline{\check{f}}^{\{\{q,r\}\}} \equiv \underline{\check{f}}^{\{\{q,r\}\}}(\underline{z})$, whose N components $f_n \equiv f_n(\underline{z})$ respectively

$\check{f}_n^{\{\{q,r\}\}} \equiv \check{f}_n^{\{\{q,r\}\}}(\underline{z})$ are the N values that the polynomials $f(z)$ respectively $\check{f}^{\{\{q,r\}\}}(z)$ take at the N (arbitrary) points z_n ,

$$f_n = f(z_n) \ , \quad \check{f}_n^{\{\{q,r\}\}} = \check{f}^{\{\{q,r\}\}}(z_n) \ , \quad n = 1, \dots, N \ . \quad (43b)$$

There holds then the N -vector formula

$$\check{\underline{f}}^{\{q^r\}} = [\check{\underline{\underline{\nabla}}}(q; \underline{z})]^r \underline{f} \ , \quad r = 0, 1, 2, \dots \ , \quad (44)$$

of course with the $(N \times N)$ -matrix $\check{\underline{\underline{\nabla}}}(q; \underline{z})$ defined componentwise by (23b). \square

Remark 2.2.2. It is again plain that the operator $\check{\nabla}(q)$, when acting on a polynomial in z of degree M , yields a polynomial of degree $M - 1$; hence, when it acts r times on any polynomial of degree less than N it yields an identically vanishing result if the integer r equals or exceeds N . Hence the right-hand side of (43a) vanishes for $r \geq N$, and this implies that the $(N \times N)$ -matrix $\check{\underline{\underline{\nabla}}}(q; \underline{z})$ features the *remarkable* property

$$[\check{\underline{\underline{\nabla}}}(q; \underline{z})]^N = \underline{\underline{0}} \ , \quad (45)$$

where again $\underline{\underline{0}}$ denotes the $(N \times N)$ -matrix with all elements vanishing. \square

The following *Proposition* and *Corollaries* are immediate consequences of these findings.

Proposition 2.2.1. Let the difference operator $\check{D}(q)$ be defined as follows,

$$\check{D}(q) = \sum_{r=0}^R \left\{ \check{d}_r(z) \ [\check{\delta}(q)]^r \right\} \ , \quad (46)$$

where the positive integer R and the $R+1$ functions $\check{d}_r(z)$ are *a priori* arbitrary (but see below the restriction on the function $f(z)$), and let

$$\check{D}(q) f(z) = g(z) \quad (47)$$

with $f(z)$ a polynomial in z of degree less than N , see (1) (but note: no such condition on $g(z)$). There then holds the N -vector equation

$$\check{\underline{\underline{D}}}(q; \underline{z}) \underline{f}(\underline{z}) = \underline{g}(\underline{z}) \quad (48)$$

with the $(N \times N)$ -matrix $\check{\underline{\underline{D}}}(q; \underline{z})$ defined as follows,

$$\check{\underline{\underline{D}}}(q; \underline{z}) = \sum_{r=0}^R \left\{ \check{d}_r(\underline{z}) \ [\check{\underline{\underline{\delta}}}(q; \underline{z})]^r \right\} \ , \quad (49)$$

of course with the N -vectors $\underline{f}(\underline{z})$ and $\underline{g}(\underline{z})$ defined as above, see (34), and the $(N \times N)$ -matrices $\check{\underline{\underline{\delta}}}(q; \underline{z})$ respectively $\check{\underline{\underline{Z}}}$ defined componentwise by (22b) respectively (11). \square

Corollary 2.2.1. If in (47) $g(z) = 0$, i. e. if for the operator $\check{D}(q)$, see (30), there holds the equation

$$\check{D}(q) f(z) = 0 \ , \quad (50a)$$

with $f(z)$ a polynomial in z of degree less than N , see (1), then the $(N \times N)$ -matrix $\underline{\underline{D}}(q; \underline{z})$, see (49), has vanishing determinant,

$$\det [\underline{\underline{D}}(q; \underline{z})] = 0 . \quad \square \quad (50b)$$

Corollary 2.2.2. If the operator $\check{D}(q)$, see (49), has the eigenvalue b ,

$$\check{D}(q) \check{f}^{(b)}(z) = b \check{f}^{(b)}(z) \quad (51a)$$

with the corresponding eigenfunction $\check{f}^{(b)}(z)$ being a polynomial in z of degree less than N , see (1), then the corresponding $(N \times N)$ -matrix $\underline{\underline{D}}(q; \underline{z})$, see (49), features the same eigenvalue b ,

$$\underline{\underline{D}}(q; \underline{z}) \underline{\check{f}}^{(b)} = b \underline{\check{f}}^{(b)} , \quad (51b)$$

and the corresponding eigenvector $\underline{\check{f}}^{(b)}$ is given by the following simple prescription,

$$\underline{\check{f}}^{(b)} = \check{f}^{(b)}(\underline{z}) \underline{u} , \quad \left(\underline{\check{f}}^{(b)} \right)_n = \check{f}^{(b)}(z_n) , \quad (51c)$$

where of course the $(N \times N)$ -matrix \underline{z} , respectively the N -vector \underline{u} , are again defined by (11) respectively (10). \square

Clearly these equations are merely examples of the neat prescriptions

$$\begin{aligned} \check{d}_s(z) &\Rightarrow \check{d}_s(\underline{z}) \underline{u} , \quad \check{\delta}(q) \Rightarrow \check{\underline{\underline{\delta}}}(q; \underline{z}) , \quad \check{\nabla}(q) \Rightarrow \check{\underline{\underline{\nabla}}}(q; \underline{z}) , \\ f(z) &\Rightarrow \underline{f}(\underline{z}) = f(\underline{z}) \underline{u} \end{aligned} \quad (52)$$

which allow to transform equations involving the action of the multiplicative operator $\check{d}_s(z)$ (see (49)) and of the operators $\check{\delta}(q)$ and $\check{\nabla}(q)$ (see (21a) and (17) or (21b)) acting on functions $f(z)$, into corresponding equations involving the action of corresponding $(N \times N)$ -matrices on corresponding N -vectors; rules that involve the introduction of N arbitrary numbers z_n (all different among themselves), and that are applicable whenever these operators act on functions $f(z)$ which are polynomials in z of degree less than the arbitrary number N , and that involve the simultaneous replacement of the function $f(z)$ into the N -vector $\underline{f}(\underline{z})$ of components $f_n(\underline{z}) = f(z_n)$.

Remark 2.2.3. An interesting generalization of all the findings reported above (in this Section 2.2) is to the case in which the function $f(z)$, instead of being a polynomial of degree less than N in z , is a polynomial of degree less than N in a variable $\zeta \equiv \zeta(z)$, see (38). It is then easily seen that all the results reported above (in this Section 2.2) remain valid, provided the $(N \times N)$ -matrix $\check{\underline{\underline{\delta}}}(q; \underline{z})$, see (22a), is replaced by the matrix $\check{\underline{\underline{\delta}}}(q; \underline{z})$ defined componentwise as follows:

$$\left(\check{\underline{\underline{\delta}}}(q; \underline{z}) \right)_{nm} = \prod_{\ell=1, \ell \neq m}^N \left[\frac{\zeta(q z_n) - \zeta_\ell}{\zeta_m - \zeta_\ell} \right] , \quad n, m = 1, \dots, N , \quad (53a)$$

where of course

$$\zeta_n \equiv \zeta(z_n) \ , \quad \underline{\zeta} = (\zeta_1, \dots, \zeta_N) \ . \quad (53b)$$

Of course likewise the matrix $\underline{\underline{\check{V}}}(q; \underline{z})$ is replaced by the matrix $\underline{\underline{\check{V}}}(q; \underline{z})$ of components

$$\begin{aligned} \left(\underline{\underline{\check{V}}}(q; \underline{z}) \right)_{nm} &= [(q-1) \ \zeta_n]^{-1} \left\{ \prod_{\ell=1, \ell \neq m}^N \left[\frac{\zeta(q z_n) - \zeta_\ell}{\zeta_m - \zeta_\ell} \right] - \delta_{nm} \right\} \ , \\ n, m &= 1, \dots, N \ ; \end{aligned} \quad (54)$$

and the N -vector $\underline{f}(\underline{z})$ of components $f_n(\underline{z}) = f(z_n)$ is replaced by the N -vector $\underline{f}(\underline{\zeta}(\underline{z}))$ of components $f_n(\underline{\zeta}(\underline{z})) = f(\zeta_n)$, while the N -vector $\underline{f}(q \underline{z})$ of components $f_n(q z_n)$ is replaced by the N -vector $\underline{f}(\underline{\zeta}(q \underline{z}))$ of components $f_n(\zeta(q z_n))$. \square

The proof of this *Remark 2.2.3* is quite analogous to that of *Lemma 2.2.1* (see Section 4) and is therefore omitted.

3 Remarkable matrices

In this Section 3—as examples of applications of the findings reported in the preceding Section 2—we identify several $(N \times N)$ -matrices which are *remarkable* because of some nontrivial properties they feature, and we report some properties of the zeros of the polynomials belonging to the Askey and q -Askey schemes.

Proposition 3.1. The $(N \times N)$ -matrix

$$\underline{\underline{\hat{K}}}(a; \underline{z}) = \underline{\underline{Z}} \ \underline{\underline{\hat{V}}}(a; \underline{z}) \ , \quad (55a)$$

hence defined componentwise as follows in terms of the $N+1$ arbitrary numbers z_n and a (see (11) and (23a)),

$$\hat{K}_{nm}(a; \underline{z}) = \left(\frac{z_n}{a} \right) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n + a - z_\ell}{z_m - z_\ell} \right) \right] - \delta_{nm} \right\} \ , \quad (55b)$$

features the N nonnegative integers less than N as its N eigenvalues:

$$\underline{\underline{\hat{K}}}(a; \underline{z}) \ \underline{v}^{(k)}(a; \underline{z}) = k \ \underline{\hat{v}}^{(k)}(a; \underline{z}) \ , \quad k = 0, 1, \dots, N-1 \ , \quad (55c)$$

$$\hat{v}_n^{(k)}(a; \underline{z}) = (z_n; a)_k = \prod_{s=0}^{k-1} (z_n + s a) \ , \quad k = 0, 1, 2, \dots, N-1 \ . \ \square \quad (55d)$$

This result is an immediate consequence of *Corollary 2.1.2*, together with the eigenvalue equation (19) and of course the definitions (11) respectively (23a) of the $(N \times N)$ -matrices $\underline{\underline{Z}}$ respectively $\underline{\underline{\hat{V}}}(a; \underline{z})$ and (19c) of the symbol $(z; a)_k$.

Remark 3.1. Note the *Diophantine* character of this result, and the fact that it implies that the $(N \times N)$ -matrix $\hat{\underline{K}}(a; \underline{z})$, which clearly depends nontrivially on the $N + 1$ numbers z_n and a , see (55b), is *isospectral* for any variation of these $N + 1$ parameters. \square

Proposition 3.2. The $(N \times N)$ -matrix

$$\hat{\underline{F}} \equiv \hat{\underline{F}}(\alpha, c; \underline{z}) = (c - \underline{z}) \hat{\underline{\delta}}(-1; \underline{z}) + (2 - \alpha) \underline{z} + (\alpha - 1) \underline{z} \hat{\underline{\delta}}(1; \underline{z}) , \quad (56a)$$

hence defined componentwise as follows in terms of the $N + 2$ arbitrary numbers z_n and α, c (see (11) and (22a)),

$$\begin{aligned} \hat{F}_{nm}(\alpha, c; \underline{z}) = & (2 - \alpha) z_n \delta_{nm} + (c - z_n) \prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n - 1 - z_\ell}{z_m - z_\ell} \right) \\ & + (\alpha - 1) z_n \prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n + 1 - z_\ell}{z_m - z_\ell} \right) , \quad n, m = 1, \dots, N , \end{aligned} \quad (56b)$$

has the N eigenvalues $c + \alpha k$ with k the first N nonnegative integers:

$$\hat{\underline{F}}(\alpha, c; \underline{z}) \underline{f}(\underline{z}, -k; c; \alpha) = (c + \alpha k) \underline{f}(\underline{z}, -k; c; \alpha) , \quad k = 0, 1, \dots, N - 1 , \quad (56c)$$

the corresponding N eigenvectors $\underline{f}(\underline{z}, -k; c; \alpha)$ being defined componentwise as follows,

$$f_n(\underline{z}, -k; c; \alpha) = F(z_n, -k; c; \alpha) , \quad (56d)$$

where $F(a, b; c; z)$ is the standard hypergeometric function (see for instance [5]),

$$F(a, b; c; z) = \sum_{r=0}^{\infty} \left[\frac{(a)_r (b)_r}{r! (c)_r} z^r \right] \quad (56e)$$

(here of course the Pochhammer symbol $(x)_r \equiv (x; 1)_r$ (see (19c)) is defined as follows:

$$(x)_0 = 1 ; \quad (x)_r = x (x + 1) \cdots (x + r - 1) , \quad r = 1, 2, \dots ; \quad (56f)$$

hence for $b = -k$ —and generic values of the parameters c and α , as we generally assume—the sum in the right-hand side of (56e) stops at $r = k$, so that $F(z, -k; c; \alpha)$ is a polynomial of degree k in z). \square

This finding is an immediate consequence of *Corollary 2.1.2* together with the formula

$$\begin{aligned} & (c - z) F(z - 1, -k; c; \alpha) + (2 - \alpha) z F(z, -k; c; \alpha) \\ & + (\alpha - 1) z F(z + 1, -k; c; \alpha) = (c + \alpha k) F(z, -k; c; \alpha) , \end{aligned} \quad (57a)$$

or equivalently (see (21a))

$$\begin{aligned} & \left[(c - z) \hat{\delta}(-1) + (2 - \alpha) z + (\alpha - 1) z \hat{\delta}(-1) \right] F(z, -k; c; \alpha) \\ & = (c + \alpha k) F(z, -k; c; \alpha) , \end{aligned} \quad (57b)$$

which coincides with eq. (2.8(28)) of [5] by replacing there z with α , a with z and moreover by setting $b = -k$ so that—when k is a nonnegative integer— $F(z, -k; c; \alpha)$ becomes a polynomial of degree k in z .

Proposition 3.3. The $(N \times N)$ -matrix

$$\underline{\hat{W}} \equiv \underline{\hat{W}}(\alpha, \beta, \gamma, \delta; \underline{z}) = B(\underline{z}) \underline{\hat{\Sigma}}(1; \underline{\zeta}) - B(-\underline{z}) \underline{\hat{\Sigma}}(-1; \underline{\zeta}) , \quad (58a)$$

hence defined componentwise as follows in terms of the $N + 4$ arbitrary numbers z_n and $\alpha, \beta, \gamma, \delta$ (see (11) and (23a)),

$$\begin{aligned} \hat{W}_{nm} = & B(z_n; \alpha, \beta, \gamma, \delta) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{(z_n + 1)^2 - z_\ell^2}{z_m^2 - z_\ell^2} \right) \right] - \delta_{nm} \right\} \\ & + B(-z_n; \alpha, \beta, \gamma, \delta) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{(z_n - 1)^2 - z_\ell^2}{z_m^2 - z_\ell^2} \right) \right] - \delta_{nm} \right\} , \end{aligned} \quad (58b)$$

with

$$B(z; \alpha, \beta, \gamma, \delta) = \frac{(z + \alpha)(z + \beta)(z + \gamma)(z + \delta)}{2z(2z + 1)} , \quad (58c)$$

features the N eigenvalues $k(k + \alpha + \beta + \gamma + \delta - 1)$ with k a nonnegative integer less than N ,

$$\begin{aligned} \underline{\hat{W}} \underline{\varphi}^{(k)}(\alpha, \beta, \gamma, \delta, z_n; \underline{z}) &= k(k + \alpha + \beta + \gamma + \delta - 1) \underline{\varphi}^{(k)}(\alpha, \beta, \gamma, \delta, z_n; \underline{z}) , \\ k &= 0, 1, \dots, N - 1 , \end{aligned} \quad (58d)$$

with the eigenvectors $\underline{\varphi}^{(k)}(\alpha, \beta, \gamma, \delta; \underline{z})$ defined componentwise as follows:

$$\varphi_n^{(k)}(\alpha, \beta, \gamma, \delta; \underline{z}) = W_k(-z_n^2; \alpha, \beta, \gamma, \delta) , \quad (58e)$$

where $W_k(\zeta; \alpha, \beta, \gamma, \delta)$ is the Wilson polynomial of degree k in ζ , defined (up to an irrelevant multiplicative constant) as follows:

$$\begin{aligned} W_k(\zeta) &\equiv W_k(\zeta; \alpha, \beta, \gamma, \delta) \\ &= \sum_{s=0}^k \left[\frac{(-k)_s (k + \alpha + \beta + \gamma + \delta - 1)_s (\alpha + \mathbf{i} z)_s (\alpha - \mathbf{i} z)_s}{s! (\alpha + \beta)_s (\alpha + \gamma)_s (\alpha + \delta)_s} \right] , \\ \zeta &= -z^2 \end{aligned} \quad (58f)$$

(see Section 1.1 of [2]; hence here $(x)_s$ is again the Pochhammer symbol, see (56f); but note that here and below the 4 parameters $\alpha, \beta, \gamma, \delta$ are *not* required to satisfy the restrictions that are instead necessary for the validity of some of the results reported in Section 1.1 of [2]; the only restrictions they must satisfy are those necessary for this definition to make good sense). \square

This finding is a direct consequence of *Corollary 2.1.2* (together with *Remark 2.1.3*), applied to the difference equation satisfied by the Wilson polynomial,

see eq. (1.1.6) of [2] (with some appropriate changes of variables, as explained in Appendix A).

Remark 3.2. Note again the *Diophantine* character of this result, and the fact that it implies that the $(N \times N)$ -matrix $\underline{\hat{W}}(\alpha, \beta, \gamma, \delta; \underline{z})$, see (58), which clearly depends nontrivially on the $N + 4$ numbers z_n and $\alpha, \beta, \gamma, \delta$, is *isospectral* for any variation of these $N + 4$ parameters which leaves invariant the sum $\alpha + \beta + \gamma + \delta$. \square

A variant of this result is provided by the following

Proposition 3.4. Let the N numbers $\tilde{\zeta}_n \equiv \tilde{\zeta}_n(\alpha, \beta, \gamma, \delta) = -\bar{z}_n^2$ be the N zeros of the Wilson polynomial $W_N(\zeta; \alpha, \beta, \gamma, \delta)$ of degree N in ζ (see Section 1.1 of [2]),

$$W_N(\zeta_n; \alpha, \beta, \gamma, \delta) = 0, \quad n = 1, \dots, N, \quad (59a)$$

and let the $(N \times N)$ -matrix $\bar{\underline{W}} \equiv \bar{\underline{W}}(\alpha, \beta, \gamma, \delta; \bar{\underline{z}})$ be defined componentwise as follows in terms of the $N + 4$ numbers \bar{z}_n and $\alpha, \beta, \gamma, \delta$,

$$\begin{aligned} \bar{W}_{nn} = & -[B(\bar{z}_n; \alpha, \beta, \gamma, \delta) + B(-\bar{z}_n; \alpha, \beta, \gamma, \delta)] \\ & + \left(\frac{4 \bar{z}_n}{2 \bar{z}_n - 1} \right) B(\bar{z}_n; \alpha, \beta, \gamma, \delta) \prod_{\ell=1, \ell \neq n}^N \left[\frac{(\bar{z}_n + 1)^2 - \bar{z}_\ell^2}{\bar{z}_n^2 - \bar{z}_\ell^2} \right], \\ n = & 1, \dots, N, \end{aligned} \quad (59b)$$

$$\begin{aligned} \bar{W}_{nm} = & B(\bar{z}_n; \alpha, \beta, \gamma, \delta) \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{(\bar{z}_n + 1)^2 - \bar{z}_\ell^2}{\bar{z}_m^2 - \bar{z}_\ell^2} \right) \right] \\ & + B(-\bar{z}_n; \alpha, \beta, \gamma, \delta) \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{(\bar{z}_n - 1)^2 - \bar{z}_\ell^2}{\bar{z}_m^2 - \bar{z}_\ell^2} \right) \right], \\ n \neq & m, \quad n, m = 1, \dots, N, \end{aligned} \quad (59c)$$

with $B(z; \alpha, \beta, \gamma, \delta)$ defined as above, see (58c). Then this matrix features the same eigenvalues and eigenvectors as the matrix $\underline{\hat{W}}(\alpha, \beta, \gamma, \delta; \underline{z})$ defined above, see *Proposition 3.3*, except that in the definition (58e) of the eigenvectors the arbitrary numbers z_n must be replaced by the N numbers \bar{z}_n such that the N numbers $\tilde{\zeta}_n = -\bar{z}_n^2$ are the N zeros of the Wilson polynomial $W_N(\zeta; \alpha, \beta, \gamma, \delta)$ of degree N in ζ (note that the matrix $\bar{\underline{W}} = \bar{\underline{W}}(\alpha, \beta, \gamma, \delta; \bar{\underline{z}})$ defined componentwise above is invariant under the exchange $\bar{z}_n \rightarrow -\bar{z}_n$, so it is in fact a function of the N numbers $\tilde{\zeta}_n \equiv \tilde{\zeta}_n(\alpha, \beta, \gamma, \delta) = -\bar{z}_n^2$ rather than the N numbers \bar{z}_n). \square

For a proof of this result see Appendix A.

Remark 3.3. This result, and analogous ones reported below (see *Propositions 3.6* and *3.9*) evoke an interesting class of *open* problems, such as that raised by the following question. Define the $(N \times N)$ -matrix $\bar{\underline{W}} \equiv \bar{\underline{W}}(\alpha, \beta, \gamma, \delta; \bar{\underline{z}})$ as in *Proposition 3.4*, but assuming that the N numbers \bar{z}_n are *a priori* arbitrary; and require that this $(N \times N)$ -matrix feature the N eigenvalues $k(k + \alpha + \beta + \gamma + \delta - 1)$ with $k = 0, 1, \dots, N - 1$. Does this requirement imply that the N numbers

$\bar{\zeta}_n = -\bar{z}_n^2$ are necessarily the N zeros of the Wilson polynomial $W_N(\zeta; \alpha, \beta, \gamma, \delta)$ of degree N in ζ ? The finding reported in [6] suggests that this is *not* the case. \square

Proposition 3.5. The $(N \times N)$ -matrix

$$\underline{\hat{R}} \equiv \underline{\hat{R}}(\alpha, \beta, \gamma, \delta; \underline{z}) = C(\underline{z}) \underline{\hat{V}}(1; \underline{\zeta}) - D(\underline{z}) \underline{\hat{V}}(-1; \underline{\zeta}) , \quad (60a)$$

hence defined componentwise as follows in terms of the $N+4$ arbitrary numbers z_n and $\alpha, \beta, \gamma, \delta$ (see (11) and (23a)),

$$\begin{aligned} \hat{R}_{nm} = C(z_n; \alpha, \beta, \gamma, \delta) & \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(z_n+1) - \zeta(z_\ell)}{\zeta(z_m) - \zeta(z_\ell)} \right) \right] - \delta_{nm} \right\} \\ + D(z_n; \alpha, \beta, \gamma, \delta) & \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(z_n-1) - \zeta(z_\ell)}{\zeta(z_m) - \zeta(z_\ell)} \right) \right] - \delta_{nm} \right\} , \end{aligned} \quad (60b)$$

with

$$\zeta(z) = z(z + \gamma + \delta + 1) \quad (60c)$$

$$C(z; \alpha, \beta, \gamma, \delta) = \frac{(z + \alpha + 1)(z + \beta + \delta + 1)(z + \gamma + 1)(z + \gamma + \delta + 1)}{(2z + \gamma + \delta + 1)(2z + \gamma + \delta + 2)} , \quad (60d)$$

$$D(z; \alpha, \beta, \gamma, \delta) = \frac{z(z - \alpha + \gamma + \delta)(z - \beta + \gamma)(z + \delta)}{(2z + \gamma + \delta)(2z + \gamma + \delta + 1)} , \quad (60e)$$

features the N eigenvalues $k(k + \alpha + \beta + 1)$ with k a nonnegative integer less than N ,

$$\begin{aligned} \underline{\hat{R}} \underline{\hat{\phi}}^{(k)}(\alpha, \beta, \gamma, \delta, z_n; \underline{z}) &= k(k + \alpha + \beta + 1) \underline{\hat{\phi}}^{(k)}(\alpha, \beta, \gamma, \delta, z_n; \underline{z}) , \\ k &= 0, 1, \dots, N-1 , \end{aligned} \quad (60f)$$

and the eigenvectors $\underline{\hat{\phi}}^{(k)}(\alpha, \beta, \gamma, \delta; \underline{z})$ defined componentwise as follows:

$$\hat{\phi}_n^{(k)}(\alpha, \beta, \gamma, \delta; \underline{z}) = R_k(\zeta(z_n); \alpha, \beta, \gamma, \delta) , \quad (60g)$$

where $R_k(\zeta; \alpha, \beta, \gamma, \delta)$ is the Racah polynomial of degree k in ζ , defined (up to an irrelevant multiplicative constant) as follows:

$$\begin{aligned} R_k(\zeta; \alpha, \beta, \gamma, \delta) &\equiv R_k(\zeta(z); \alpha, \beta, \gamma, \delta) \\ &= \sum_{s=0}^k \left[\frac{(-k)_s (k + \alpha + \beta + 1)_s (-z)_s (z + \gamma + \delta + 1)_s}{s! (\alpha + 1)_s (\beta + \delta + 1)_s (\gamma + 1)_s} \right] \end{aligned} \quad (60h)$$

(see Section 1.2 of [2]; hence here $(x)_s$ is again the Pochhammer symbol, see (56f); but note that here and below neither the 4 parameters $\alpha, \beta, \gamma, \delta$ nor the argument ζ are required to satisfy the restrictions that are instead necessary for the validity of most of the results reported in Section 1.2 of [2]; the only

restrictions they must satisfy are those necessary for this definition to make good sense). \square

This finding is a direct consequence of *Corollary 2.1.2* (together with *Remark 2.1.3*), applied to the difference equation satisfied by the Racah polynomial, see eq. (1.2.5) of [2]; its proof is omitted because it is analogous—*mutatis mutandis*—to the proof (given in Appendix A) of *Proposition 3.3*.

Remark 3.4. Note again the *Diophantine* character of this result, and the fact that it implies that the $(N \times N)$ -matrix $\underline{\hat{R}}(\alpha, \beta, \gamma, \delta; \underline{z})$, see (60), which clearly depends nontrivially on the $N + 4$ numbers z_n and $\alpha, \beta, \gamma, \delta$, is *isospectral* for any variations of these $N + 4$ parameters which leaves invariant the sum $\alpha + \beta$. \square

Proposition 3.6. Let the N numbers $\bar{\zeta}_n \equiv \bar{\zeta}_n(\alpha, \beta, \gamma, \delta) = \bar{z}_n (\bar{z}_n + \gamma + \delta + 1)$ be the N zeros of the Racah polynomial $R_N(\zeta; \alpha, \beta, \gamma, \delta)$ of degree N in ζ (see Section 1.2 of [2]),

$$R_N(\bar{\zeta}_n; \alpha, \beta, \gamma, \delta) = 0, \quad n = 1, \dots, N, \quad (61a)$$

and let the $(N \times N)$ -matrix $\bar{\underline{R}} \equiv \bar{\underline{R}}(\alpha, \beta, \gamma, \delta; \bar{z})$ be defined componentwise as follows in terms of the $N + 4$ numbers \bar{z}_n and $\alpha, \beta, \gamma, \delta$:

$$\begin{aligned} \bar{R}_{nn} &= -[C(\bar{z}_n) + D(\bar{z}_n)] + 2 \left(\frac{2 \bar{z}_n + \gamma + \delta + 1}{2 \bar{z}_n + \gamma + \delta} \right) \cdot C(\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N \left[\frac{\zeta(\bar{z}_n + 1) - \bar{\zeta}_\ell}{\zeta(\bar{z}_n - 1) - \bar{\zeta}_\ell} \right], \\ n &= 1, \dots, N, \end{aligned} \quad (61b)$$

$$\begin{aligned} \bar{R}_{nm} &= C(\bar{z}_n; \alpha, \beta, \gamma, \delta) \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(\bar{z}_n + 1) - \zeta(\bar{z}_\ell)}{\zeta(\bar{z}_m) - \zeta(\bar{z}_\ell)} \right) \right] \\ &+ D(\bar{z}_n; \alpha, \beta, \gamma, \delta) \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(\bar{z}_n - 1) - \zeta(\bar{z}_\ell)}{\zeta(\bar{z}_m) - \zeta(\bar{z}_\ell)} \right) \right], \end{aligned} \quad (61c)$$

$$n \neq m, \quad n, m = 1, \dots, N, \quad (61d)$$

where of course $\zeta(z) = z(z + \gamma + \delta + 1)$, see (60c), and we omitted to indicate explicitly the dependence of $C(\bar{z}_n)$ and $D(\bar{z}_n)$, see (60d) and (60e), on the 4 parameters $\alpha, \beta, \gamma, \delta$. Then this matrix features the same eigenvalues and eigenvectors as the matrix $\underline{\hat{R}}(\alpha, \beta, \gamma, \delta; \underline{z})$ defined above, see *Proposition 3.1.5*, except that in the definition (60g) of the eigenvectors the arbitrary numbers z_n must be replaced by the N numbers \bar{z}_n such that the N numbers $\bar{\zeta}_n \equiv \bar{\zeta}_n(\alpha, \beta, \gamma, \delta) = \bar{z}_n (\bar{z}_n + \gamma + \delta + 1)$ are the N zeros of the Racah polynomial $R_N(\zeta; \alpha, \beta, \gamma, \delta)$ of degree N in ζ . \square

Again, the proof of this result is omitted because it is analogous—*mutatis mutandis*—to the proof of *Proposition 3.4* given in Appendix A.

Remark 3.5. Since the Wilson and the Racah polynomials are the "highest" polynomials belonging to the Askey scheme (see for instance [2]), analogous results involving all the "lower" polynomials of the Askey scheme can be obtained from those reported above—see *Propositions 3.3, 3.4, 3.5* and *3.6*—by appropriate reductions. \square

Proposition 3.7. The $(N \times N)$ -matrix

$$\underline{\underline{\tilde{K}}}(q; \underline{z}) = \underline{\underline{Z}} \underline{\underline{\tilde{V}}}(q; \underline{z}) , \quad (62a)$$

hence defined componentwise as follows in terms of the $N + 1$ arbitrary numbers z_n and q (see (11) and (23b)),

$$\tilde{K}_{nm}(q; \underline{z}) = \left(\frac{1}{q-1} \right) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{q z_n - z_\ell}{z_m - z_\ell} \right) \right] - \delta_{nm} \right\} , \quad (62b)$$

features the N eigenvalues $(1-q)^k / (1-q)$ with k the N nonnegative integers less than N :

$$\underline{\underline{\tilde{K}}}(q; \underline{z}) \underline{\underline{\tilde{v}}}^{(k)}(q; \underline{z}) = \left(\frac{1-q^k}{1-q} \right) \underline{\underline{\tilde{v}}}^{(k)}(q; \underline{z}) , \quad (62c)$$

$$\tilde{v}_n^{(k)}(a; \underline{z}) = (z_n)^k , \quad k = 0, 1, \dots, N-1 . \quad \square \quad (62d)$$

This result is an immediate consequence of *Corollary 2.1.2*, together with the eigenvalue equation (20a) and of course the definitions (11) respectively (23b) of the $(N \times N)$ -matrix $\underline{\underline{Z}}$ respectively $\underline{\underline{\tilde{V}}}(q; \underline{z})$.

Remark 3.6. Note again the *Diophantine* character of this result, and the fact that it implies that the $(N \times N)$ -matrix $\underline{\underline{\tilde{K}}}(q; \underline{z})$, which clearly depends nontrivially on the $N + 1$ numbers z_n and q , see (62b), is *isospectral* for any variations of the N parameters z_n . \square

Proposition 3.8. The $(N \times N)$ -matrix $\underline{\underline{\tilde{Y}}}$ defined componentwise as follows in terms of the $N + 5$ arbitrary numbers z_n and $\alpha, \beta, \gamma, \delta, q$,

$$\begin{aligned} \tilde{Y}_{nm} = & A(z_n; \alpha, \beta, \gamma, \delta; q) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(q z_n) - \zeta(z_\ell)}{\zeta(z_m) - \zeta(z_\ell)} \right) \right] - \delta_{nm} \right\} \\ & + A(z_n^{-1}; \alpha, \beta, \gamma, \delta; q) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(q^{-1} z_n) - \zeta(z_\ell)}{\zeta(z_m) - \zeta(z_\ell)} \right) \right] - \delta_{nm} \right\} , \end{aligned} \quad (63a)$$

with

$$\zeta \equiv \zeta(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad (63b)$$

and

$$A(z; \alpha, \beta, \gamma, \delta; q) = \frac{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)}{(1-z^2)(1-q z^2)} , \quad (63c)$$

features the N eigenvalues $(q^{-k} - 1)(1 - \alpha\beta\gamma\delta q^{k-1})$ with k a nonnegative integer less than N ,

$$\begin{aligned} \underline{\check{Y}} \underline{\check{\varphi}}^{(k)}(\alpha, \beta, \gamma, \delta; q; \underline{z}) &= (q^{-k} - 1)(1 - \alpha\beta\gamma\delta q^{k-1}) \underline{\check{\varphi}}^{(k)}(\alpha, \beta, \gamma, \delta; q; \underline{z}) , \\ k &= 0, 1, \dots, N-1 , \end{aligned} \quad (63d)$$

with the eigenvectors $\underline{\check{\varphi}}^{(k)}(\alpha, \beta, \gamma, \delta; q; \underline{z})$ defined componentwise as follows:

$$\check{\varphi}_n^{(k)}(\alpha, \beta, \gamma, \delta; q; z_n) = p_k(\zeta_n; \alpha, \beta, \gamma, \delta; q; \zeta_n) , \quad (63e)$$

where $p_k(\alpha, \beta, \gamma, \delta; q; \zeta)$ is the Askey-Wilson polynomial of degree k in ζ , defined (up to an irrelevant multiplicative constant; see eq. (3.1.1) of [2]) as follows:

$$\begin{aligned} &p_k(\alpha, \beta, \gamma, \delta; q; \zeta) \\ &= \sum_{s=0}^k \left[\frac{(q^{-k}; q)_s (\alpha\beta\gamma\delta q^{k-1}; q)_s [\alpha, \zeta; q]_s q^s}{(q; q)_s (\alpha\beta; q)_s (\alpha\gamma; q)_s (\alpha\delta; q)_s} \right] , \\ &\zeta = \frac{1}{2} \left(z + \frac{1}{z} \right) , \quad k = 0, 1, 2, \dots , \end{aligned} \quad (63f)$$

where $(x; q)_s$ is the " q -shifted factorial",

$$(x; q)_0 = 1 ; \quad (x; q)_s = (1-x)(1-qx)(1-q^2x)\dots(1-q^{s-1}x) , \quad s = 1, 2, \dots \quad (63g)$$

and

$$[\alpha, \zeta; q]_s = (1 - \alpha z; q)_s \left(1 - \frac{\alpha}{z}; q \right)_s = \prod_{r=0}^{s-1} (1 - 2\alpha\zeta q^r + \alpha^2 q^{2r}) \quad (63h)$$

(see Sections 0.1 and 3.1 of [2]; but note that here and below the 4 parameters $\alpha, \beta, \gamma, \delta$ are *not* required to satisfy the restrictions that are instead necessary for the validity of some of the results reported in Section 3.1 of [2]; the only restrictions they must satisfy are those necessary for this definition to make good sense). \square

This finding is a direct consequence of *Corollary 2.1.2* (together with *Remark 2.1.3*), applied to the difference equation satisfied by the Askey-Wilson polynomial reading (see eq. (3.1.7) of [2], with some appropriate notational changes)

$$\begin{aligned} &A(z) [p_k(\zeta(qz)) - p_k(\zeta)] + A(z^{-1}) \left[p_k\left(\zeta\left(\frac{z}{q}\right)\right) - p_k(\zeta) \right] \\ &= (q^{-k} - 1)(1 - \alpha\beta\gamma\delta q^{k-1}) p_k(\zeta) , \end{aligned} \quad (64)$$

where of course $A(z) \equiv A(z; \alpha, \beta, \gamma, \delta; q)$ and $p_k(\zeta) \equiv p_k(\alpha, \beta, \gamma, \delta; q; \zeta)$ are defined as above, see (63c) and (63f), and—most importantly— $\zeta \equiv \zeta(z)$, see (63b).

Remark 3.7. Note again the *Diophantine* character of this result, and the fact that it implies that the $(N \times N)$ -matrix $\underline{\underline{Y}} \equiv \underline{\underline{Y}}(\alpha, \beta, \gamma, \delta; q; \underline{z})$, which clearly depends nontrivially on the $N + 5$ numbers $z_n, \alpha, \beta, \gamma, \delta$ and q , see (63), is *isospectral* for any variations of the $N + 4$ parameters $z_n, \alpha, \beta, \gamma, \delta$ which keeps constant the product $\alpha\beta\gamma\delta$. \square

Proposition 3.9. Let the N numbers $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$ be such that the N numbers $\bar{\zeta}_n \equiv \bar{\zeta}_n(\alpha, \beta, \gamma, \delta; q; N) = (\bar{z}_n + 1/\bar{z}_n)/2$ are the N zeros of the Askey-Wilson polynomial $p_N(\alpha, \beta, \gamma, \delta; q; \zeta)$ of degree N in ζ (see (63f)),

$$p_N(\alpha, \beta, \gamma, \delta; q; \bar{\zeta}_n) = 0, \quad n = 1, \dots, N, \quad (65a)$$

so that, up to an irrelevant multiplicative constant,

$$p_N(\alpha, \beta, \gamma, \delta; q; \zeta) = \prod_{k=1}^N (\zeta - \bar{\zeta}_k); \quad (65b)$$

and let the $(N \times N)$ -matrix $\bar{\underline{\underline{Y}}} \equiv \bar{\underline{\underline{Y}}}(\alpha, \beta, \gamma, \delta; q; \bar{\underline{z}})$ be defined componentwise as follows in terms of the $N + 5$ numbers \bar{z}_n and $\alpha, \beta, \gamma, \delta, q$,

$$\begin{aligned} \bar{Y}_{nn} = & - \left[A(\bar{z}_n) + A\left(\frac{1}{\bar{z}_n}\right) \right] \\ & + (1+q) \left(\frac{\bar{z}_n^2 - 1}{\bar{z}_n^2 - q} \right) A(\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N \left[\frac{\zeta(q \bar{z}_n) - \zeta(\bar{z}_\ell)}{\zeta(\bar{z}_m) - \zeta(\bar{z}_\ell)} \right] \Bigg\}, \\ n = & 1, \dots, N, \end{aligned} \quad (65c)$$

$$\begin{aligned} \bar{Y}_{nm} = & A(\bar{z}_n) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(q \bar{z}_n) - \zeta(\bar{z}_\ell)}{\zeta(\bar{z}_m) - \zeta(\bar{z}_\ell)} \right) \right] \right\} \\ & + A\left(\frac{1}{\bar{z}_n}\right) \left\{ \left[\prod_{\ell=1, \ell \neq m}^N \left(\frac{\zeta(q^{-1} \bar{z}_n) - \zeta(\bar{z}_\ell)}{\zeta(\bar{z}_m) - \zeta(\bar{z}_\ell)} \right) \right] \right\}, \\ n \neq & m, \quad n, m = 1, \dots, N, \end{aligned} \quad (65d)$$

of course with $A(z) \equiv A(z; \alpha, \beta, \gamma, \delta; q)$ respectively $\zeta(z)$ defined as above, see (63c) respectively (63b). Then this matrix features the same eigenvalues and eigenvectors as the matrix $\underline{\underline{Y}} \equiv \underline{\underline{Y}}(\alpha, \beta, \gamma, \delta; q; \underline{z})$ defined above, see *Proposition 3.8*, except that in the definition (63e) of the eigenvectors the N arbitrary numbers z_n must be replaced by the N numbers \bar{z}_n . \square

The proof of this result is analogous to that of *Propositions 3.4* and *3.6*; it takes advantage of the relation

$$\begin{aligned} & A\left(\frac{1}{\bar{z}_n}\right) \prod_{\ell=1, \ell \neq n}^N \left[\zeta\left(\frac{\bar{z}_n}{q}\right) - \zeta(\bar{z}_\ell) \right] \\ = & \left(\frac{q \bar{z}_n^2 - 1}{\bar{z}_n^2 - q} \right) A(\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N [\zeta(q \bar{z}_n) - \zeta(\bar{z}_\ell)] \end{aligned} \quad (66)$$

easily seen to be implied by (64) together with (65a) and (65b).

Remark 3.8. Analogous results to those reported in *Propositions 3.8* and *3.9* but involving the q -Racah instead of the Askey-Wilson polynomials can be obtained in an analogous manner, or via the relation among the polynomials of these two classes reported at the end of Section 3.1 of [2]. Moreover, since the Askey-Wilson and the q -Racah polynomials are the "highest" polynomials belonging to the q -Askey scheme (see for instance [2]), analogous results involving all the "lower" polynomials of the q -Askey scheme can be obtained from those reported above by appropriate reductions. \square

Remark 3.9. The results reported above involving $(N \times N)$ -matrices constructed with polynomials belonging to the Askey and q -Askey schemes have already been obtained—up to notational changes, and some unessential restrictions on their validity—by Ryu Sasaki (private communication, and see [7]); for analogous results see [8]. Other recent papers reporting somewhat analogous results for the zeros of named polynomials are listed (with no pretence to completeness) in Ref. [9]. \square

Remark 3.10. Let us finally emphasize that the $(N \times N)$ -matrices $\hat{\underline{\underline{\delta}}}(a)$ respectively $\check{\underline{\underline{\delta}}}(q)$ defined componentwise by (22a) respectively (22b) are themselves quite *remarkable*, see below the two *Remarks 4.1.1* and *4.2.1* and Appendix B. And the $(N \times N)$ -matrices $\hat{\underline{\underline{\nabla}}}(a; \underline{z})$ respectively $\check{\underline{\underline{\nabla}}}(q; \underline{z})$ also feature remarkable properties, see (29) respectively (45). \square

4 Proofs of the main results

In this Section 4 we provide the missing proofs of the findings reported in Section 2. We use of course the notation introduced above, see Sections 1 and 2. The alert and informed reader will note the analogy of these proofs to that of (8) provided in Section 2.4 of [1]; indeed the starting point are the Lagrangian interpolational formulas (2) and (3), applicable to any function $f(z)$ which is a polynomial in z of degree less than N .

4.1 The $(N \times N)$ -matrices $\underline{\underline{Z}}$, $\hat{\underline{\underline{\delta}}}(a; \underline{z})$ and $\hat{\underline{\underline{\nabla}}}(a; \underline{z})$

It is plain, see (2) and (3), that

$$f(z+a) = \sum_{m=1}^N \left[f_m p_{N-1}^{(m)}(z+a) \right] \quad (67)$$

and that

$$p_{N-1}^{(m)}(z+a) = \prod_{\ell=1, \ell \neq m}^N \left(\frac{z+a-z_\ell}{z_m-z_\ell} \right), \quad m=1, \dots, N. \quad (68)$$

Hence

$$f(z+a) = \sum_{m=1}^N \left[f_m \prod_{\ell=1, \ell \neq m}^N \left(\frac{z+a-z_\ell}{z_m-z_\ell} \right) \right] , \quad (69)$$

implying, for $z = z_n$,

$$f(z_n+a) = \sum_{m=1}^N \left[f_m \prod_{\ell=1, \ell \neq m}^N \left(\frac{z_n+a-z_\ell}{z_m-z_\ell} \right) \right] . \quad (70)$$

One therefore concludes that the N -vector $\underline{f}(z+a)$, of components $f_n(z+a) = f(z_n+a)$, is given by the N -vector formula

$$\underline{f}(z+a) = \hat{\underline{\underline{\delta}}}(a; \underline{z}) \underline{f}(z) , \quad (71a)$$

with the $(N \times N)$ -matrix $\hat{\underline{\underline{\delta}}}(a; \underline{z})$ defined by (22a). Iterating r times this formula implies

$$\underline{f}(z+r a) = \left[\hat{\underline{\underline{\delta}}}(a; \underline{z}) \right]^r \underline{f}(z) . \quad (71b)$$

This coincides with (26), proving *Lemma 2.1.1*.

Remark 4.1.1. Note that the formulas (71) clearly imply the matrix identity

$$\hat{\underline{\underline{\delta}}}(r a; \underline{z}) = \left[\hat{\underline{\underline{\delta}}}(a; \underline{z}) \right]^r , \quad r = 0, 1, 2, \dots . \quad (72)$$

Indeed, more generally, the formulas (71) clearly imply

$$\begin{aligned} \underline{f}(z+a+b) &= \hat{\underline{\underline{\delta}}}(a+b; \underline{z}) \underline{f}(\underline{z}) \\ &= \hat{\underline{\underline{\delta}}}(a; \underline{z}) \hat{\underline{\underline{\delta}}}(b; \underline{z}) \underline{f}(z) = \hat{\underline{\underline{\delta}}}(a; \underline{z}) \hat{\underline{\underline{\delta}}}(b; \underline{z}) \underline{f}(z) , \end{aligned} \quad (73a)$$

entailing the *remarkable* matrix identities

$$\hat{\underline{\underline{\delta}}}(a; \underline{z}) \hat{\underline{\underline{\delta}}}(b; \underline{z}) = \hat{\underline{\underline{\delta}}}(b; \underline{z}) \hat{\underline{\underline{\delta}}}(a; \underline{z}) = \hat{\underline{\underline{\delta}}}(a+b; \underline{z}) . \quad (73b)$$

And these formulas, together with the obvious identity $\hat{\underline{\underline{\delta}}}(0; \underline{z}) = \underline{I}$, with \underline{I} the $(N \times N)$ unit matrix, also imply the *remarkable* matrix identity

$$\left[\hat{\underline{\underline{\delta}}}(a; \underline{z}) \right]^{-1} = \hat{\underline{\underline{\delta}}}(-a; \underline{z}) . \quad \square \quad (73c)$$

4.2 The $(N \times N)$ -matrices $\underline{\underline{Z}}$, $\check{\underline{\underline{\delta}}}(q; \underline{z})$ and $\check{\underline{\underline{\nabla}}}(q; \underline{z})$

The treatment in this Section 4.2 is analogous—*mutatis mutandis*—to that of the previous Section 4.1; hence its presentation is a bit more terse.

The starting point are again the basic formulas of Lagrangian interpolation, see (2) and (3). They clearly imply

$$f(qz) = \sum_{m=1}^N \left[f_m p_{N-1}^{(m)}(qz) \right], \quad (74)$$

and

$$p_{N-1}^{(m)}(q, z) = \prod_{\ell=1, \ell \neq m}^N \left(\frac{q z - z_\ell}{z_m - z_\ell} \right), \quad m = 1, \dots, N; \quad (75)$$

hence

$$f(qz) = \sum_{m=1}^N \left[f_m \prod_{\ell=1, \ell \neq m}^N \left(\frac{qz - z_\ell}{z_m - z_\ell} \right) \right], \quad (76)$$

implying, for $z = z_n$,

$$f(q, z_n) = \sum_{m=1}^N \left[f_m \prod_{\ell=1, \ell \neq m}^N \left(\frac{q, z_n - z_\ell}{z_m - z_\ell} \right) \right]. \quad (77)$$

One therefore concludes that the N -vector $\underline{f}(q\ z)$, of components $f_n(q\ z) = f(q\ z_n)$, is given by the N -vector formula

$$\underline{f}(q\,z) = \underline{\check{\delta}}(q;\underline{z})\,\underline{f}(z)\;, \quad (78a)$$

with the $(N \times N)$ -matrix $\underline{\underline{\delta}}(a; \underline{z})$ defined by (22b). Iterating r times this formula one gets

$$\underline{f}(q^r z) = [\underline{\tilde{\delta}}(q; \underline{z})]^r \underline{f}(z) \ , \quad r = 1, 2, \dots \ . \quad (78b)$$

This coincides with (26), proving *Lemma 2.2.1*.

Remark 4.2.1. Note that the formulas (78) clearly imply the matrix identity

$$\check{\underline{\underline{\Delta}}}(r^q; \underline{z}) = [\check{\underline{\underline{\Delta}}}(q; \underline{z})]^r, \quad r = 0, 1, 2, \dots. \quad (78c)$$

Indeed, more generally, the formulas (78) clearly imply

[illegible]

entailing the *remarkable* matrix identities

$$\check{\underline{\underline{\delta}}}(p; \underline{z}) \check{\underline{\underline{\delta}}}(q; \underline{z}) = \check{\underline{\underline{\delta}}}(q; \underline{z}) \check{\underline{\underline{\delta}}}(p; \underline{z}) = \check{\underline{\underline{\delta}}}(pq; \underline{z}) \quad . \quad (79b)$$

And these formulas, together with the obvious identity $\check{\underline{\delta}}(1; \underline{z}) = \underline{I}$, with \underline{I} the $(N \times N)$ unit matrix, also imply the remarkable identity

$$[\check{\underline{\underline{\delta}}}(q; \underline{z})]^{-1} = \check{\underline{\underline{\delta}}}(q^{-1}; \underline{z}) \ . \ \square \quad (79c)$$

5 Outlook

In this last section we mention tersely possible future developments.

A possibility is to investigate finite-dimensional representations of difference operators which are *exact* in other functional spaces: for instance in functional spaces spanned by other seed functions than polynomials (of degree less than N), possibly also including functions of more than a single dependent variable. For previous developments in these directions see for instance [10], [1].

Another possibility is to identify many more *remarkable* matrices than the representative examples reported in Section 3; such as those discussed in the various subsections of Section 2.4.5 (entitled "Remarkable matrices and identities") and in Appendix D (entitled "Remarkable matrices and related identities") of [1], and in other papers referred to there.

Yet another possibility is to identify *solvable* nonlinear dynamical systems, for instance by extending to exact finite-dimensional representations of *difference* operators the techniques based on exact finite-dimensional representations of *differential* operators, as described in Section 2.5 (entitled "Many-body problems on the line solvable via techniques of exact Lagrangian interpolation") of [1].

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7 Appendix A

In this Appendix A we tersely outline the derivation of *Propositions 3.3* and *3.4*.

The starting point to prove *Propositions 3.3* is the difference equation satisfied by the Wilson polynomial $W_k(\zeta; \alpha, \beta, \gamma, \delta)$ of degree k in ζ (see eq. (1.1.6) of [2]; and note that the validity of this relation for Wilson polynomials does not require any limitation on the 4 parameters $\alpha, \beta, \gamma, \delta$ other than those required in order that their definition (58f) make good sense). We conveniently write this difference equation as follows, via some trivial notational changes (including the relations $x = \mathbf{i}z$ and $\zeta = x^2 = -z^2$, and the omission of the explicit indication

of dependence on the 4 parameters $\alpha, \beta, \gamma, \delta$:

$$\begin{aligned} & \left[B(z) \hat{\nabla}(1) - B(-z) \hat{\nabla}(-1) \right] W_k(-z^2) \\ &= k(k + \alpha + \beta + \gamma + \delta - 1) W_k(-z^2) , \quad k = 0, 1, 2, \dots , \end{aligned} \quad (80)$$

with $B(z)$ defined as above, see (58c), and the operators $\hat{\nabla}(\pm 1)$ acting on functions of the variable z as follows (see (16)):

$$\hat{\nabla}(\pm 1) f(z) = \pm [f(z \pm 1) - f(z)] . \quad (81)$$

It is plain that the operator

$$\hat{D} = B(z) \hat{\nabla}(1) - B(-z) \hat{\nabla}(-1) , \quad (82)$$

when acting on functions of the variable $\zeta = -z^2$, yields functions of this variable ζ (not of z), hence we can consider the equation (80) as an eigenvalue equation,

$$\hat{D} W_k(\zeta) = k(k + \alpha + \beta + \gamma + \delta - 1) W_k(\zeta) , \quad (83)$$

satisfied by polynomials $W_k(\zeta)$ of degree k in the variable ζ .

The validity of *Proposition 3.3* is then an immediate consequence of *Corollary 2.1.2*, together with *Remark 2.1.3*.

Moreover, let the N zeros of the polynomial $W_N(-z^2) \equiv W_N(\zeta)$ be denoted as $\bar{\zeta}_j = -\bar{z}_j^2$, $j = 1, \dots, N$, so that (up to an irrelevant multiplicative constant)

$$W_N(-z^2) = \prod_{j=1}^N (-z^2 + \bar{z}_j^2) = \prod_{j=1}^N (\zeta - \bar{\zeta}_j) . \quad (84)$$

Then (80) with $k = N$ and $z = \bar{z}_n$, $n = 1, \dots, N$, implies

$$\begin{aligned} & B(\bar{z}_n) \prod_{j=1}^N [(\bar{z}_n + 1)^2 - \bar{z}_j^2] + B(-\bar{z}_n) \prod_{j=1}^N [(\bar{z}_n - 1)^2 - \bar{z}_j^2] = 0 , \\ & n = 1, \dots, N , \end{aligned} \quad (85a)$$

entailing

$$\begin{aligned} & (2\bar{z}_n + 1) B(\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N [(\bar{z}_n + 1)^2 - \bar{z}_\ell^2] \\ & + (-2\bar{z}_n + 1) B(-\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N [(\bar{z}_n - 1)^2 - \bar{z}_\ell^2] = 0 \end{aligned} \quad (85b)$$

hence

$$\begin{aligned} & B(-\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N [(\bar{z}_n - 1)^2 - \bar{z}_\ell^2] = \left(\frac{2\bar{z}_n + 1}{2\bar{z}_n - 1} \right) \cdot \\ & \cdot B(\bar{z}_n) \prod_{\ell=1, \ell \neq n}^N [(\bar{z}_n + 1)^2 - \bar{z}_\ell^2] , \quad n = 1, \dots, N . \end{aligned}$$

Now one observes that *Proposition 3.3* holds for *any* arbitrary assignment of the N numbers z_n ; hence it holds in particular for the assignment $z_n = \bar{z}_n$. And with this assignment—via the last formula written above—*Proposition 3.3* clearly becomes *Proposition 3.4*, which is thereby proven.

8 Appendix B

In this Appendix B we list two *remarkable* identities satisfied by the two $(N \times N)$ -matrices $\hat{\underline{\underline{\delta}}}(a; \underline{z})$ and $\check{\underline{\underline{\delta}}}(q; \underline{z})$. Their proof is an immediate consequence of the identities satisfied by the two corresponding operators $\hat{\delta}(a)$ and $\check{\delta}(q)$, the validity of which is quite obvious.

$$\check{\underline{\underline{\delta}}}(q^{-1}; \underline{z}) \hat{\underline{\underline{\delta}}}(a; \underline{z}) \check{\underline{\underline{\delta}}}(q; \underline{z}) = \hat{\underline{\underline{\delta}}}(q a; \underline{z}) ; \quad (86)$$

$$\check{\underline{\underline{\delta}}}(q^{-1}; \underline{z}) \hat{\underline{\underline{\delta}}}(a; \underline{z}) \check{\underline{\underline{\delta}}}(q; \underline{z}) \hat{\underline{\underline{\delta}}}(b; \underline{z}) = \hat{\underline{\underline{\delta}}}(a + q^{-1} b; \underline{z}) . \quad (87)$$

Other analogous identities can be obviously obtained.

It is moreover plain that there holds the matrix-vector eigenvalue equation

$$\check{\underline{\underline{\delta}}}(q; \underline{z}) (\underline{\underline{Z}} \underline{u}) = q^k (\underline{\underline{Z}} \underline{u}) , \quad k = 0, 1, \dots, N-1 , \quad (88a)$$

implying

$$\det [\check{\underline{\underline{\delta}}}(q; \underline{z})] = q^{N(N-1)/2} , \quad \text{trace} [\check{\underline{\underline{\delta}}}(q; \underline{z})] = \frac{1 - q^N}{1 - q} . \quad (88b)$$

Here of course the two $(N \times N)$ -matrices $\hat{\underline{\underline{\delta}}}(a; \underline{z})$ and $\check{\underline{\underline{\delta}}}(q; \underline{z})$ are defined componentwise by (22) in terms of the N components z_n of the N -vector \underline{z} (which are N *arbitrary* numbers, except for the restriction to be all different among themselves), and of the arbitrary parameters a and q ; while $\underline{\underline{Z}} = \text{diag}[z_n]$ and the N -vector \underline{u} has all components equal to unity, $\underline{u} = (1, 1, \dots, N)$.

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